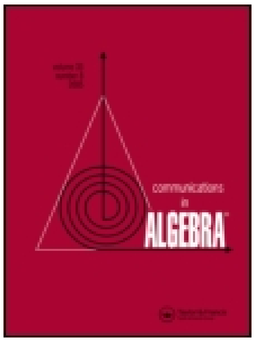


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# A LIE–RINEHART ALGEBRA WITH NO ANTIPODE

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***The aim of this note is to communicate a simple example of a Lie–Rinehart algebra whose enveloping algebra is not a Hopf algebroid, neither in the sense of Böhm and Szlachányi, nor in the sense of Lu.***

*Key Words:* Hopf algebroids; Lie algebroids; Lie–Rinehart algebras.

*2010 Mathematics Subject Classification:* 16T05; 57T05; 16S30.



## 1. INTRODUCTION

The enveloping algebra of a Lie algebra is a classical example of a Hopf algebra. Hence it is natural to ask whether the enveloping algebra of a Lie algebroid [12] or more generally of a Lie–Rinehart algebra [13] carries the structure of a Hopf algebroid. It turns out that they always are *left bialgebroids* (introduced under the name $\times_R$-bialgebras by Takeuchi [15]), see [16], and in fact *left Hopf algebroids* (introduced under the name $\times_R$-Hopf algebras by Schauenburg [14]), see [6, Example 2]; see also [5, 11].

However, there is also the definition of a Hopf algebroid due to Lu [9] and the one due to Böhm and Szlachányi [2] (the latter will be called *full Hopf algebroids* from now on), which both assume the existence of an antipode satisfying certain axioms. The aim of the present paper is to communicate a concrete example of a Lie–Rinehart algebra whose universal enveloping algebra is not a Hopf algebroid in either of these two settings.

This clarifies further the relation between the three concepts: it is well-known and easily seen that every full Hopf algebroid is a left Hopf algebroid, see [2]. In [1], an example of a full Hopf algebroid was given that is not a Hopf algebroid in the sense of [9], but to the best of our knowledge, it is not known whether every Hopf algebroid in the sense of Lu satisfies the axioms of a full or at least of a left Hopf algebroid, and whether every left Hopf algebroid admits an antipode satisfying


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either of the definitions of [2, 9] (a counterexample announced in [8, Remark 3.12] did not appear in print).

In the light of [8, Proposition 3.11], it is known that the enveloping algebras of Lie algebroids [3] and of the Lie–Rinehart algebras associated to Poisson algebras [4, Section (3.2)] are full Hopf algebroids. However, here we prove the following theorem.

**Theorem 1.1.** *Let $K$ be a field, $R := K[x, y]/\langle x \cdot y, x^2, y^2\rangle$, $L$ be the 1-dimensional Lie algebra with basis $\{\alpha\}$, and $E \in \mathrm{Der}_K(R)$ be the derivation with $E(x) = y$, $E(y) = 0$.*

1. *There is a Lie–Rinehart algebra structure on $(R, L)$ with $R$-module structure on $L$ given by $x \cdot \alpha = y \cdot \alpha = 0$ and anchor map given by $\rho(\alpha) = E$.*
2. *There is no right $V(R, L)$-module structure on $R$ that extends multiplication in $R$.*
3. *$V(R, L)$ is neither a full Hopf algebroid, nor a Hopf algebroid in the sense of* [9].

The note is structured as follows. In Section 2, we recall some basic definitions and prove the implication 2. $\Rightarrow$ 3. of Theorem 1.1. In Section 3, we provide a construction method of Lie–Rinehart algebras whose enveloping algebras satisfy part 2, for which the Lie–Rinehart algebra from 1 is a basic example.

## 2. BACKGROUND

This section contains background on Lie–Rinehart algebras [13], see also [4, 7, 8, 11] for more information. For the corresponding differential geometric notion of a Lie algebroid see [12] and for example [10] for further details.

We fix a field $K$. An unadorned $\otimes$ denotes the tensor product of $K$-vector spaces.

**Definition 2.1.** A Lie–Rinehart algebra consists of the following elements:

1. A commutative $K$-algebra $(R, \cdot)$;
2. A Lie algebra $(L, [-, -]_L)$ over $K$;
3. A left $R$-module structure $R \otimes L \to L$, $r \otimes \xi \mapsto r \cdot \xi$, $r \in R$, $\xi \in L$; and
4. An $R$-linear Lie algebra homomorphism $\rho : L \to \mathrm{Der}_K(R)$ satisfying

$$[\xi, r \cdot \zeta]_L = r \cdot [\xi, \zeta]_L + \rho(\xi)(r) \cdot \zeta, \quad r \in R, \xi, \zeta \in L. \tag{2.1}$$

The map $\rho$ is referred to as the anchor map.

There are two fundamental examples: if $R$ is any commutative algebra, one can take $L$ to be $\mathrm{Der}_K(R)$ with its usual Lie algebra and $R$-module structure, and $\rho = \mathrm{id}$. The other extreme is $R = K$ and $\rho = 0$, $L$ being any Lie algebra.

In his paper [13], Rinehart generalised the construction of the universal enveloping algebra of a Lie algebra to Lie–Rinehart algebras, see Section 2 therein for the precise construction. The result is an associative $K$-algebra $V(R, L)$ that is generated by the (sum of the) images of a $K$-algebra map

$$R \longrightarrow V(R, L)$$

and a Lie algebra map

$$(L, [-, -]_L) \longrightarrow (V(R, L), [-, -]), \quad \xi \longmapsto \bar{\xi},$$

where $[-, -]$ denotes the commutator in $V(R, L)$. As Rinehart, we do not distinguish between an element in $R$ and its image in $V(R, L)$ which is justified as the first map is always injective. The construction is such that in $V(R, L)$ one has for all $r \in R$, $\xi \in L$

$$[\bar{\xi}, r] = \rho(\xi)(r), \quad r\bar{\xi} = \overline{r \cdot \xi}, \tag{2.2}$$

where the product in $V(R, L)$ is denoted by concatenation.

As indicated in the introduction, $V(R, L)$ has the structure of a left Hopf algebroid. Its counit endows $R$ with the structure of a left $V(R, L)$-module, in such a way that the induced action of $r \in R$ is given by multiplication, and the induced action of $\xi \in L$ is given by the anchor map. The following fact is well known and yields the implication 2. ⇒ 3. in Theorem 1.1.

**Lemma 2.2.** *If $H$ is either a full Hopf algebroid or a Hopf algebroid in the sense of Lu, with antipode $S : H \to H$, left counit $\varepsilon : H \to R$, and source and target maps $s, t : R \to H$, then defining for $h \in H$, $r \in R$*

$$rh := \varepsilon(S(h)s(r)) \tag{2.3}$$

*yields a right $H$-module structure on $R$ for which the underlying left $R$-action on $R$ is given by left multiplication.*

*Proof.* The canonical left action of a left bialgebroid $H$ on the base algebra $R$ is given by $hr := \varepsilon(hs(r)) = \varepsilon(ht(r))$, and the antipode of a Hopf algebroid is an algebra antihomomorphism (see [1, Proposition 4.4] respectively [9, Definition 4.1] for the two different notions). Hence (2.3) defines a right action of $H$ on $R$. Finally, one has $S \circ t = s$ (see [1, Definition 4.1 (iii)] respectively [9, Definition 4.1.2.]), so $rt(q) = \varepsilon(S(t(q))s(r)) = \varepsilon(s(q)s(r)) = qr$ for all $q, r \in R$. □

## 3. A LIE–RINEHART ALGEBRA WITHOUT FLAT RIGHT CONNECTION ON *R*

We now prove Theorem 1.1, 1. and 2. We begin by considering more generally Lie–Rinehart algebras $(R, L)$ whose $R$-module structure on $L$ is given by a character $\chi : R \to K$.

**Lemma 3.1.** *Let $(R, \cdot)$ be a commutative $K$-algebra, $(L, [-, -]_L)$ be a Lie algebra, and $\rho : L \to \mathrm{Der}_K(R)$ be a Lie algebra map. Define an $R$-module structure on $L$ by $r \cdot \xi := \chi(r)\xi$, where $\chi : R \to K$ is a character on $R$. Then $(R, L)$ is a Lie–Rinehart algebra if and only if $\rho$ is $R$-linear and $\rho(\xi)(r) \in \ker \chi$ for all $r \in R$, $\xi \in L$.*

*Proof.* This follows as the Leibniz rule (2.1) takes the form

$$[\xi, \chi(r)\zeta]_L = \chi(r)[\xi, \zeta]_L + \chi(\rho(\xi)(r))\zeta$$

and hence by the $K$-linearity of the bracket becomes equivalent to $\rho(\xi)(r) \in \ker\chi$. □

Note that for these examples, $[-.-]_L$ is even $R$-linear, so $L$ is a Lie algebra over $R$. However, in general we have $\rho \neq 0$.

Assume now that $(R, L)$ is a Lie–Rinehart algebra as in the above lemma, and that multiplication in $R$ can be extended to a right $V(R, L)$-module structure on $R$. Denote by $\partial(\xi) \in R$ the element obtained by acting with $\xi \in L$ on $1 \in R$ under this right action. This defines a $K$-linear map $\partial : L \to R$, and in $V(R, L)$ we have

$$\rho(\xi)(r) = [\bar{\xi}, r] = \bar{\xi}r - r\bar{\xi} = \bar{\xi}r - \overline{r \cdot \xi} = \bar{\xi}r - \chi(r)\bar{\xi},$$

so by acting with this element on $1 \in R$, one sees that this map $\partial$ satisfies

$$\rho(\xi)(r) = \partial(\xi) \cdot (r - \chi(r)). \tag{3.1}$$

A $K$-linear map $\partial$ with this property defines a right $V(R, L)$-module structure extending multiplication on $R$ if and only if it satisfies the condition $\partial([\xi, \zeta]_L) = \rho(\xi)(\partial(\zeta)) - \rho(\zeta)(\partial(\xi))$. It also corresponds to a generator of the Gerstenhaber bracket on $\Lambda_R L$, see [4], but we shall not need these facts.

*Proof of Theorem 1.1, 1. and 2.* The first part is verified by explicit computation; the Lie–Rinehart algebra is of the form as in Lemma 3.1 with $\chi$ given by $\chi(x) = \chi(y) = 0$.

For 2., take $r = x$ and $\xi = \alpha$ in (3.1). One obtains $y = E(x) = \rho(\alpha)(x) = \partial(\alpha) \cdot x$. However, there is no element $z \in R$ such that $y = z \cdot x$. □

Carrying out Rinehart's construction explicitly yields a presentation of the associative $K$-algebra $V(R, L)$ in terms of generators $x, y, \bar{\alpha}$ satisfying the relations

$$\bar{\alpha}x = y, \quad \bar{\alpha}y = x\bar{\alpha} = y\bar{\alpha} = x^2 = y^2 = xy = yx = 0.$$

Hence $V(R, L)$ has a $K$-linear basis given by $\{\bar{\alpha}^n, x, y\}_{n \in \mathbb{N}}$. The source and target maps are both the inclusion of $R$ into $V(R, L)$. Hence one can also see directly that $V(R, L)$ admits no antipode: $S$ would satisfy $S(x) = x$, $S(y) = y$ and one would have $y = S(y) = S(\bar{\alpha}x) = S(x)S(\bar{\alpha}) = xS(\bar{\alpha})$, but there is no element $z \in V(R, L)$ such that $y = xz$.

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